MATH 301: INTRODUCTION TO PROOFS
HOMEWORK 5

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Problem 1. Let $f\colon X \to Y$ be a function.

(i) Show that the assignment which assigns $f(x)$ to $(x, f(x))$ defines a function from $\overline{\pi}_2\colon \mathbf{Gr}(f) \to \mathbf{Im}(f)$ which is surjective.

(ii) Show that the following diagram of functions commute:

$$\begin{tikzcd}
\mathbf{Gr}(f) \arrow[r, tail] \arrow[d, two heads, "\overline{\pi}_2"'] & X \times Y \arrow[d, "\pi_2"] \\
\mathbf{Im}(f) \arrow[r, tail] & Y
\end{tikzcd}$$

(iii) What is the fibre of an element $y \in \mathbf{Im}(f)$ along the function $\overline{\pi}_2$? Supply a proof for your answer.

Problem 2. In the previous lecture, we proved that any isomorphism of sets is a bijection. Does the converse hold as well? If yes, give a proof, and if no, supply a counter-exmaple.

Problem 3. For each of the following pairs of sets X, Y, construct a bijection $f\colon X \to Y$. In each case include your proof that f is bijective.

(i) $X = \mathbb{Z}$ and $Y = \mathbb{N}$

(ii) $X = \mathbb{R}$ and $Y = (-1, 1)$

Problem 4. Prove that multiplication of natural numbers is commutative, that is for all natural numbers m and n we have $m \cdot n = n \cdot m$.

Problem 5. Suppose you have an infinite chessboard with a natural number written in each square. The value in each square is the average of the values of the four neighboring squares. Prove that all the values on the chessboard are equal.

Problem 6. In the lecture on isomorphisms, we defined the set $\mathbb{B}^+_\infty$ of monotone infinite binary numbers. We showed that $\mathbb{B}^+_\infty$ is isomorphic to, and hence in bijection with, the set $\mathbb{N}_\infty = \{0, 1, 2, \ldots, \infty\}$ of extended natural numbers.[1]

(i) Show that the function $\mathsf{pred}\colon \mathbb{N}_\infty \to 1 + \mathbb{N}_\infty$ defined by

$$\mathsf{pred}(x) = \begin{cases} \mathsf{inl}(*) & \text{if } x = \mathsf{0} \\ \mathsf{inr}(n) & \text{if } x = \mathsf{succ}\, n \\ \mathsf{inr}(\infty) & \text{if } x = \infty \end{cases}$$

is a bijection. Hence,

$$\mathbb{N}_\infty \cong 1 + \mathbb{N}_\infty\,.$$

(ii) Use the isomorphism $\mathbb{B}^+_\infty \cong \mathbb{N}_\infty$ to obtain a function $\overrightarrow{\mathsf{pred}}\colon \mathbb{B}^+_\infty \to 1 + \mathbb{B}^+_\infty$. Show that $\overrightarrow{\mathsf{pred}}$ is a bijection. What is the result of applying $\overrightarrow{\mathsf{pred}}$ to an arbitrary element of $\mathbb{B}^+_\infty$? In particular, which element of $1 + \mathbb{B}^+_\infty$ equals $\overrightarrow{\mathsf{pred}}(\overline{1})$ where $\overline{1}$ is the infinite sequence which consists solely of 1's.

[1] Here we write 0 for $\mathsf{0}$, and 1 for $\mathsf{succ}\,\mathsf{0}$, and 2 for $\mathsf{succ}\,\mathsf{succ}\mathsf{0}$ and so on.