MATH 301: INTRODUCTION TO PROOFS
HOMEWORK 6

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Problem 1.

(i) Show that if $f\colon X \to X$ is idempotent, then $\mathrm{Fix}(f) \cong \mathrm{Im}(f)$.

(ii) For an idempotent function $f\colon X \to X$, show that

$$X/\sim_f \;\cong\; \mathrm{Fix}(f) \cong \mathrm{Im}(f)$$

(iii) Using the previous parts, construct an idempotent function $f\colon \mathbb{N}\times\mathbb{N} \to \mathbb{N}\times\mathbb{N}$ such that $\mathrm{Fix}(f)$ is in bijection with the set of integers. Describe the elements of $\mathrm{Fix}(f)$.

Problem 2. Suppose $r\colon A \to B$ is a retraction. Show that B is in bijection with the quotient $A/\sim_r$.

Problem 3. In the previous lecture, we defined the set $\mathbb{Q}$ of rational numbers to be the quotient set by the equivalence relation

$$(u, a) \approx (v, b) =_{\text{def}} (u(b + 1) = v(a + 1))$$

on the set $\mathbb{Z}\times\mathbb{N}$. Define the relation

$$(u, a) \approx' (v, b) \Leftrightarrow ub = va$$

on the set $\mathbb{Z}\times\mathbb{Z}\setminus\{0\}$.

(i) Show that this relation is an equivalence relation.

(ii) Write $\mathbb{Q}'$ for the quotient set obtained by the relation $\approx'$. Show that $\mathbb{Q}'$ is in bijection with $\mathbb{Q}$, and conclude that it either way we define the same set of rational numbers.

Problem 4.

(i) Show that, for all $x, y \in \mathbb{R}_{\mathsf{d}}$,

$$\neg(x < y) \Rightarrow y \leqslant x$$

(ii) Use the previous result to prove that, assuming Excluded Middle, we have that for all $x, y \in \mathbb{R}_{\mathsf{d}}$,

$$(x < y) \vee (y \leqslant x)$$

(iii) Does $\neg(x \leqslant y)$ imply $y < x$?