MATH 301

INTRODUCTION TO PROOFS

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- What are proofs?
- Why proofs?
- Examples of proofs

Overview

1. Introduction
2. Two examples of mathematical proofs

What is a proof?

You are in Amsterdam in the year 2021 and you want to visit both Rembrandt Museum and Van Gogh Museum.

Single Canal, Amsterdam - Image by: Koen Smilde

What is a proof?

Statement: There is a path from Rembrandt Museum to van Gogh Museum crossing exactly six bridges.

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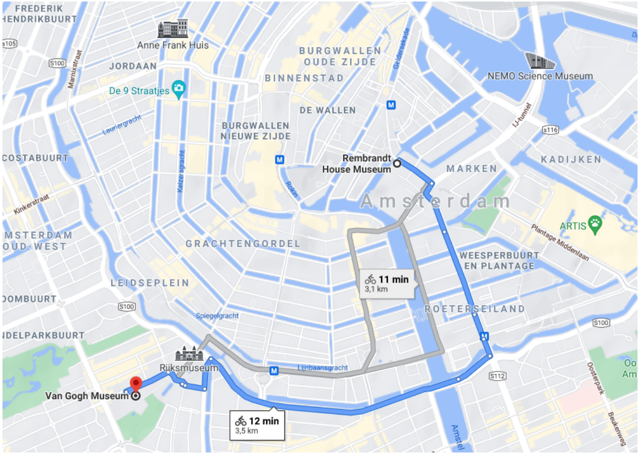
Proof:



Source: Google Maps

Why proofs?

- Mathematical proofs have two purposes:
  - to convince oneself and others of truth of various statements,
  - and to convey mathematical ideas and methods.
- In the second and third lectures we will focus on the first purpose by giving precise rules for writing proofs.

Overview

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We give many proofs of the proposition above. Below is the first proof:

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$$
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0 &\leqslant (x - \frac{1}{x})^2 \\
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&= x^2 + \frac{1}{x^2} - 2
\end{aligned}
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Let a be a positive real number. There is some real x such that $x^2 = a$. Hence

$$a + \frac{1}{a} = x^2 + \frac{1}{x^2} \geqslant 2\,.$$

□

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Corollary.

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Theorem (AM-GM inequality for two real variables)

For two non-negative real numbers x and y,

$$\sqrt{xy} \leqslant \frac{x+y}{2}$$

Corollary.

If a is positive real number then $a + \frac{1}{a} \geqslant 2$.

Proof of corollary.

Set $x = a$ and $y = \frac{1}{a}$ in the theorem above. $\square$

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Suppose x and y are non-negative real numbers. Since xy is non-negative, $\sqrt{xy} \leqslant \frac{x+y}{2}$ if and only if $xy \leqslant \left(\frac{x+y}{2}\right)^2$. The latter holds if and only if $4xy \leqslant (x+y)^2$.

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$$(x+y)^2 - 4xy = x^2 + y^2 + 2xy - 4xy = (x-y)^2 \geqslant 0\,.$$

□

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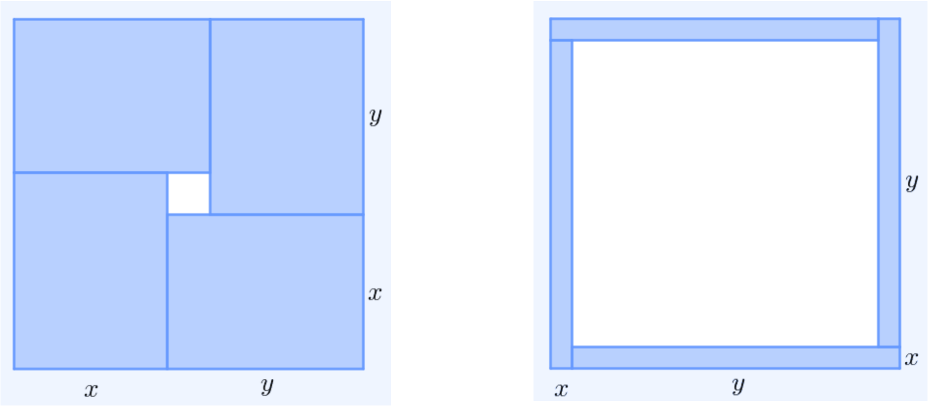

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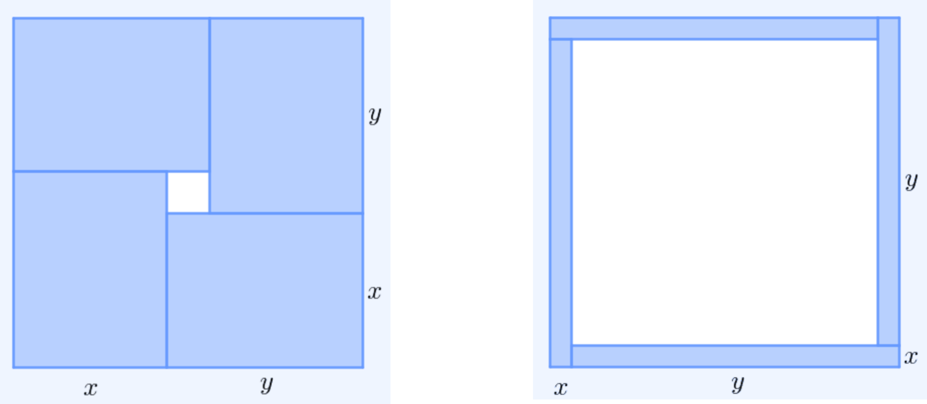
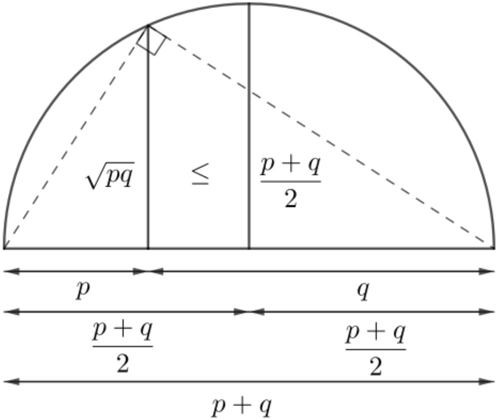


Question: Does a geometric illustration/ explanation count as a proof?

A direct geometric proof of AM-GM inequality



Source: Wikipedia

A proof using calculus

Let a be a positive number. Hence we can find a real number t such that $a = e^t$.

A proof using calculus

Let a be a positive number. Hence we can find a real number t such that $a = e^t$. Therefore $a + \frac{1}{a} = e^t + e^-t$. Let $f(t) = e^t + e^-t$. Note that f is a function of $t \in \mathbb{R}$ and is symmetric about the y-axis, that is $f(t) = f(-t)$. Note also that $f'(t) = e^t - e^{-t}$ which is positive for all $t \geqslant 0$. Therefore $f(t)$ is increasing for $t \geqslant 0$ and decreasing for $t \leqslant 0$ due to its symmetry about the y-axis. Hence the minimum of $f(t)$ occurs at $t = 0$. Therefore, the minimum of $a + \frac{1}{a}$ occurs at $a = e^0 = 1$. Therefore, $a + \frac{1}{a} \geqslant 2$.

Theorem (J.J. Sylvester)

A finite collection $\mathcal{P}$ of points in the plane has the property that any line through two of them passes through a third. Show that all the points in $\mathcal{P}$ lie on a line.

Sources: JHU graphic and pictorial collection

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Either $\mathcal{P}$ is empty or there is a point in $\mathcal{P}$. If $\mathcal{P}$ is empty then the statement holds vacuously.Suppose $\mathcal{P}$ is non-empty. Suppose the points in $\mathcal{P}$ are not colinear. Among pairs (P, ℓ) consisting of a line ℓ, passing through two different points of $\mathcal{P}$, and a point P of $\mathcal{P}$ not on that line, choose one, say (P_0, ℓ_0), which minimizes the distance d from P to ℓ.

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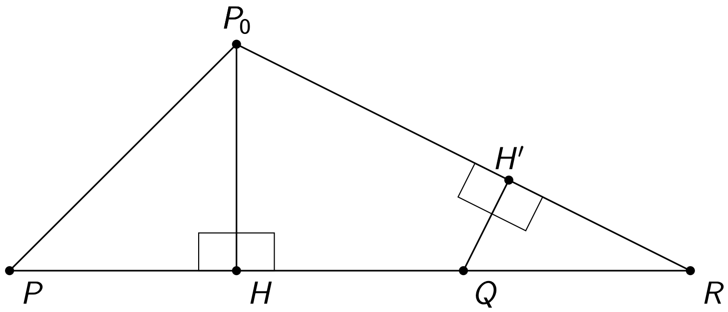
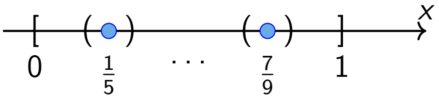


P0
H′
P
H
Q
R

Later in the course we will see more complicated theorems and proofs where our intuition and what proofs say begin to diverge.

Theorem

For any positive real ϵ, there is a collection $(U_n \mid n \in \mathbb{N})$ of open intervals such that together they cover all the rational numbers between 0 and 1 and the sum of the length of these intervals is less than ϵ.



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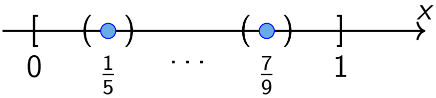

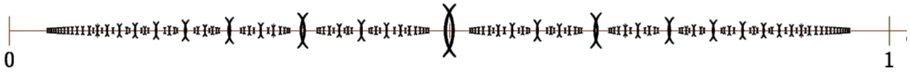

The End

Thanks for your attention!