MATH 301

INTRODUCTION TO PROOFS

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- Propositional Logic
- Natural Deduction

Overview

In this lecture

In this lecture, we shall introduce propositional logic and we shall use it to analyze the forms of mathematical reasoning.

Symbolic logic

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- In a similar way, symbolic logic is an idealized model of mathematical language and proof.
- Curiously, mathematicians did not really study the proofs that they were constructing until the 20th century. Once they did, they discovered that logic itself was a deep topic with many implications for the rest of mathematics.

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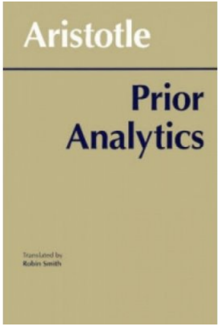

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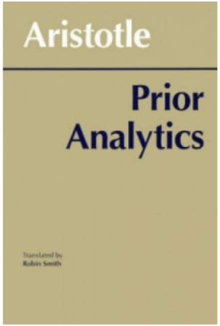
- The crucial observation of Aristotle: the correctness of inferences of statements has nothing to do with the content, truth or falsity of the individual statements, but, rather, the general patterns/rules of reasoning.



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- He showed us that we can classify valid patterns of inference by their logical form.

Examples of syllogism

Every man is an animal.
Every animal is mortal.
Therefore every man is mortal.

Every A is B.
Every B is C.
Therefore every A is C.

$$\frac{BaA \quad CaB}{CaA}$$

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Every human is mortal.
No cyborg is mortal.
Therefore no cyborg is a human.

Every N is M.
No X is M.
Therefore no X is N.

$$\frac{MaN \quad MeX}{NeX}$$

Historical context

First, to say about what and of what this is an investigation: it is about demonstration and of demonstrative science. Then, to define what is a premise, what is a term, and what a syllogism, and which kind of syllogism is perfect and which imperfect. [...] A premise, then, is a sentence that affirms or denies something of something, and this is either universal or particular or indeterminate. By 'universal' I mean belonging to all or to none of something; by 'particular' , belonging to some, or not to some, or not to all ; by 'indeterminate', belonging without universality or particularity, as in 'of contraries there is a single science' or 'pleasure is not a good'.

(Aristotle, Prior Analytics Book I, translated by Gisela Striker)

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- Expressions like ‘because’, ‘since’, and ‘after all’ are used to indicate that the claims that follow are the premises of the argument.

Overview

New propositions from the old

- We can make the following new propositions from propositions P and Q.

Proposition	Notation
P and Q	$P \wedge Q$
P or Q	$P \vee Q$
P implies Q	$P \Rightarrow Q$
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- Therefore, if $P : \mathbb{P}\text{rop}$ and $Q : \mathbb{P}\text{rop}$ then $P \wedge Q : \mathbb{P}\text{rop}$, $P \vee Q : \mathbb{P}\text{rop}$, $P \Rightarrow Q : \mathbb{P}\text{rop}$, $P \Leftrightarrow Q : \mathbb{P}\text{rop}$, $\neg P : \mathbb{P}\text{rop}$, $\neg Q : \mathbb{P}\text{rop}$, etc.

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- An inference is valid if it can be justified by fundamental rules of reasoning that reflect the meaning of the logical terms involved.
- In natural deduction, every proof is a proof from hypotheses. In other words, in any proof, there is a finite collection of hypotheses $P_1, P_2, \ldots, P_n$ and a conclusion Q, and the proof shows that how Q follows from $P_1, P_2, \ldots, P_n$.

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A lazy (but wrong) way of writing

$$(P_1 \Rightarrow P_2) \wedge (P_2 \Rightarrow P_3) \wedge \ldots \wedge (P_{n-1} \Rightarrow P_n)\,. \tag{1}$$

is

$$P_1 \Rightarrow P_2 \Rightarrow P_3 \Rightarrow \ldots \Rightarrow P_n\,.$$

However since the expression (1) is tedious as it repeats the propositions $P_2, \ldots, P_{n-1}$, we allow for the short hand notation

$$\begin{aligned} & P_1 \\ \Rightarrow\ & P_2 \\ \Rightarrow\ & \ldots \\ \Rightarrow\ & P_n \end{aligned}$$

to denote that the chain of implications (1) leads to the conclusion $P_1 \Rightarrow P_n$.

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We show that

$$(P \Rightarrow (Q \Rightarrow R)) \Rightarrow (P \wedge Q \Rightarrow R)$$

is a tautology.

$$\dfrac{\dfrac{[P \wedge Q]^{1}}{Q}}{}$$

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