First order logic

1. In the previous lectures we learned about propositional logic, i.e. the rules governing reasoning about and with propositions.

propositional variables

→ complex proposition

→ rules of inference

2) This logic has some serious limitations that will be overcome by First Order Logic (FOL)

Example:

1) No child is older than his or her parents.

2) If someone is alone, they are not with someone else.

3) There is no greatest odd integer.

4) There is an integer divisible by all other integers.

3. We need a way to talk about

- objects / individuals
- properties of objects
- relationship between objects

4. To this end, we need to introduce

$x, y, z, \ldots$ • variables

$\forall, \exists$ • quantification (over variables)

$\dfrac{?}{?}$ • the inference rules for quantifiers

5 Variables.

Consider the assertions

(i) The integer n is divisible by 7.

(ii) $x^2 = 1 \Rightarrow (x = 1) \vee (x = -1)$

Where does x come from? is it an integer, or a real number?

(iii) If m is an integer which when divided by 4 leaves the remainder 3, then it has a prime factor which also leaves the remainder 3 when divided by 4.

- None of the assertions are propositions (why?)
- The truth of each proposition depends on the value of variables involved in it (e.g. n, x, m)

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The integer n is divisible by 7.

Predicate

$x^2 = 1$

$x = 1$

$x = -1$

m leaves the remainder 3 when divided by 4

variable

n, x, m

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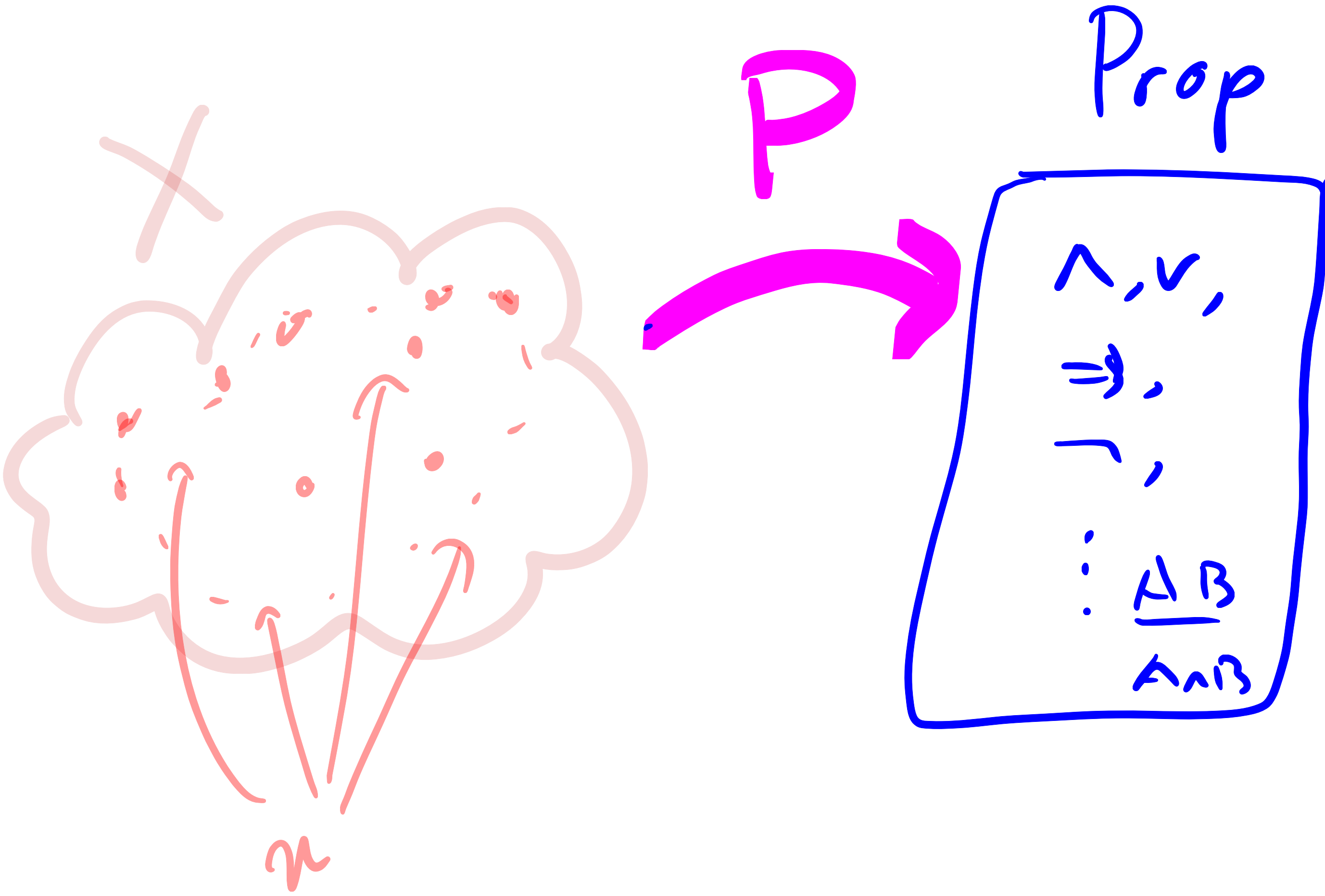
The domain of a predicate $P(x)$ is the domain where the variable x vary over:



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Quantifiers are the right tools in our logical toolbox which give us the ability to write down general/universal assertions in symbolic logic.

Example

ex1. No child is older than his or her parents.

$\forall x. \forall y.$
$(C(x,y) \Rightarrow \neg O(x,y))$

$C(x,y)$ = x is a child of y

$P(x,y)$ = x is a parent of y

$O(x,y)$ = x is older than y

or

$\nexists x \nexists y.$
$C(x,y) \wedge O(x,y)$

ex2. There is no greatest odd integer.

$int(x)$ = x is an integer.

$leq(x,y)$ = x is less than or equal to y

$$\nexists x.\ (int(x) \wedge \forall y.\ (int(y) \Rightarrow leq(y,x)))$$

Note If we know the domain of interpretation of the variables in our logical assertions, then we can use more familiar symbols to encode our mathematical assertions into FoL.

$$\nexists x.\ (\mathrm{int}(x) \wedge \forall y.\ (\mathrm{int}(y) \Rightarrow \mathrm{leq}(y,x)))$$

vs

$$\nexists x.\ \forall y.\ (y \leq x)$$

or more explicitly

$$\nexists x \in \mathbb{Z}.\ \forall y \in \mathbb{Z}.\ (y \leq x)$$

⑨ Free & bound variables.

bound to $\forall$

$$\forall x.\ A(x)$$

$$\forall x.\ A(x) \vee B(x)$$

$$\forall x.\ (\mathrm{even}(x) \vee \mathrm{odd}(x))$$

$$\forall x.\ A(x) \wedge C((x, y))$$

free

The statement

$$\forall x.\ A(x)$$

is not at all about x.

e.g $\forall x.\ \mathrm{even}(x) \vee \mathrm{odd}(x)$

says

every natural number is either even or odd;

<u>no reference to x.</u>

(i)

$$\forall x \; (\text{even}(x) \lor \text{odd}(x)) \equiv$$

$$\forall y \; (\text{even}(y) \lor \text{odd}(y))$$

(ii) $\forall y \; (x \leq y) \equiv \forall z \; (x \leq z)$

(iii) $\underbrace{\forall y (x \leq y)}_{\text{statement about } x} \not\equiv \forall y (w \leq y)$

statement about w