ex3.

(i) $(\exists x. ♥(you, x)) \land (\exists y. ♥(y, you))$

the scope of x

the scope of y

(ii) $\exists x. ♥(you, x) \land (\exists x. ♥(x, you))$

(iii) $\exists x. (♥(you, x) \land ♥(x, you))$ $\Rightarrow$ you are happy

the scope of x

(i) $\equiv$ (ii) $\not\equiv$ (iii)

④ Free & bound variables.

bound to $\forall$

$$\forall x.\ A(x)$$

$$\forall x.\ A(x) \lor B(x)$$

$$\forall x.\ (even(x) \lor odd(x))$$

$$\forall x.\ A(x) \land C(x, y)$$

free

Ex. $P(x,y) :=$ x is a parent of y.

$C(x,y) :=$ x and y are a couple.

Consider the following statements.

(i) $\exists z\ P(x,z) \Rightarrow \exists y\ C(x,y)$

(ii) $\neg\left(C(x,y) \Rightarrow \exists z\ P(x,z) \wedge P(y,z)\right)$

In (i) $\begin{cases} \text{free vars: } x \\ \text{bounded vars: } y, z \end{cases}$

In (ii) $\begin{cases} \text{free vars: } x, y \\ \text{bounded vars: } z \end{cases}$

Ex. In the statement "for every integer n, there is a prime number p between n and $2n$"

n is a bound variable and
p is a free variable.

Formally,

$$\forall n \ \exists p \left(\text{Prime}(p) \wedge (n \leq p) \wedge (p \leq 2n) \right)$$

Ex. In the statement

$$a \geq 0 \Rightarrow \exists b \ (a = b^2)$$

a is free and b is bounded.

The statement

$$\forall x.\ A(x)$$

is not at all about x.

e.g $\forall x.\ \text{even}(x) \vee \text{odd}(x)$

says

every natural number is either even or odd;

no reference to x.

4½ Change of bound variables

(i)

$$\forall x\ (\text{even}(x) \vee \text{odd}(x)) \equiv$$

$$\forall y\ (\text{even}(y) \vee \text{odd}(y))$$

(ii) $\forall y\ (x \leq y) \equiv \forall z\ (x \leq z)$

⚠ z must be a fresh variable.

e.g.

$$\forall y\ (x \leq y) \not\equiv \forall x.\ (x \leq x)$$

(iii) $\underbrace{\forall y\ (x \leq y)}_{\text{statement about } x} \not\equiv \underbrace{\forall y\ (w \leq y)}_{\text{statement about } w}$

⚠ substitution (i.e. change) of a free variable by a fresh variable results in different formula.

$$P(x) \rightsquigarrow P(x)[w/x] \equiv P(w)$$

Binding priorities

Earlier in the course, we learned about the binding priorities of propositions:

(i) $\neg$ (ii) $\wedge, \vee$ (iii) $\Rightarrow$

Now, we add quantifier in between:

(i) $\neg$ (ii) $\forall, \exists$ (iii) $\wedge, \vee$, (iv) $\Rightarrow$

For instance the expression $\exists x.\ A(x) \wedge B(x)$ is parsed as $\left(\exists x.\ A(x)\right) \wedge B(x)$.

Example. Parse the following expressions by inserting brackets following the binding convention:

$$\exists x\, D(x) \Rightarrow \forall y\, D(y)$$

Answer:

$$\left(\exists x\, D(x)\right) \Rightarrow \left(\forall y\, D(y)\right) \qquad (1)$$

which is different from

$$\exists x\, \left(D(x) \Rightarrow \forall y\, D(y)\right) \qquad (2)$$

(1) is true if the universe of discourse is empty whereas (2) is false in that case

How to prove a universally quantified statement

Recall that in order to prove that $\sqrt{2}$ is not rational we used the lemma

$$\forall a \, \forall b \, (b \neq 0) \Rightarrow \neg (a^2 = 2b^2)$$

To prove the latter we start by letting

a, b to be arbitrary integers.

Here is how the proof of irrationality of $\sqrt{2}$ goes:

Let a and b be arbitrary integers.
Suppose $b \neq 0$, and suppose $a^2 = 2b^2$

$\vdots$

Contradiction. $\square$

Here's the last proof presented in natural deduction.

$$
\dfrac{\dfrac{\dfrac{\dfrac{\begin{matrix}\overline{b \neq 0}^{\,1} & \qquad & \overline{a^2 = 2b^2}^{\,2} \\ \vdots & & \vdots \end{matrix}}{\perp}\,2}{\neg(a^2 = 2b^2)}\,1}{b \neq 0 \Rightarrow \neg(a^2 = 2b^2)}}{\forall b\, (b \neq 0 \Rightarrow \neg(a^2 = 2b^2))}
$$

$$
\overline{\forall a\, \forall b\, (b \neq 0 \Rightarrow \neg(a^2 = 2b^2))}
$$

intro rule for $\forall$

$$\frac{A(x)}{\forall x . A(x)}\ (\forall \text{ intro})$$

⚠ x should not be free in any hypothesis which has not been cancelled!

e.g.

$$\frac{\begin{array}{c}\overline{P(x)}^{1} \\ \vdots \\ Q(x)\end{array}}{\forall x . Q(x)}$$

x free in $P(x)$

Not allowed!

Elim rule for $\forall$

$$\frac{\forall x. A(x)}{A(t)}\ (\forall\text{elim})$$

t : an arbitrary term (?)

$$\frac{\forall x\, \exists y\ \ y > x}{\exists y.\, y > y+1}\quad (x = y+1)$$

truth is not preserved!
what did we do wrong?

Example. We construct a natural deduction proof of

$$\forall x\, A(x) \Rightarrow \forall x\, B(x) \Rightarrow \forall y\,(A(y) \wedge B(y))$$

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\overline{\forall x\, A(x)}^{\,1}}{A(y)}\,(\forall E) \qquad \dfrac{\overline{\forall x\, B(x)}^{\,2}}{B(y)}\,(\forall E)}{A(y) \wedge B(y)}\,(\wedge I)}{\forall y\,(A(y) \wedge B(y))}}{\forall x\, B(x) \to \forall y\,(A(y) \wedge B(y))}\,2}{\forall x\, A(x) \to \forall x\, B(x) \to \forall y\,(A(y) \wedge B(y))}\,1$$

Ex. In a town there is a barber that shaves all and only the men who do not shave themselves. Show that this is a contradiction.

Define $S(x,y) =$ x shaves y.

$$\dfrac{\dfrac{}{S(b,b)\lor\neg S(b,b)}\,(LEM) \qquad \dfrac{\dfrac{}{S(b,b)}\,1 \qquad \dfrac{\dfrac{}{S(b,b)}\,1 \quad \dfrac{\forall x.(S(b,x)\Leftrightarrow\neg S(x,x))}{S(b,b)\Leftrightarrow\neg S(b,b)}}{\neg S(b,b)}}{\bot} \qquad \dfrac{\dfrac{}{\neg S(b,b)}\,2 \qquad \dfrac{\dfrac{}{\neg S(b,b)}\,1 \quad \dfrac{\forall x.(S(b,x)\Leftrightarrow\neg S(x,x))}{S(b,b)\Leftrightarrow\neg S(b,b)}}{S(b,b)}}{\bot}}{\bot}\,1$$

<u>Ex.</u> Suppose E and O are predicates with one variable ranging over natural numbers.

Suppose also that

(i) $\forall n. (\neg E(n) \Rightarrow O(n))$

Prove that $\forall n.\ O(n) \vee E(n)$

$$\dfrac{\dfrac{E(n) \vee \neg E(n) \qquad \dfrac{\overline{E(n)}^{\,1}}{O(n) \vee E(n)} \qquad \dfrac{\dfrac{\dfrac{\forall n. (\neg E(n) \to O(n))}{\neg E(n) \to O(n)} \qquad \overline{\neg E(n)}^{\,1}}{O(n)}}{O(n) \vee E(n)}}{O(n) \vee E(n)}}{\forall n.\ O(n) \vee E(n)}$$

Intro rule for $\exists$

$$\frac{A(t)}{\exists x \; A(x)}$$

Not quite correct!

$$\frac{\forall y \; (y=y)}{\exists x \; \forall y \; (y=x)}$$

A wrong inference

Caveat. The term t in $A(t)$ should not clash with bound variables in A.

Elim rule for $\exists$

$$\dfrac{\exists x\, A(x) \qquad \begin{matrix} \overline{A(y)}^{\,1} \\ \vdots \\ B \end{matrix}}{B}$$

Note: y should not be free in B.

Ex. Let's prove

$$\exists x \, (A(x) \vee B(x)) \Rightarrow \exists x A(x) \vee \exists x B(x)$$

$$\dfrac{\dfrac{}{\exists x \, (A(x) \vee B(x))}1 \qquad \dfrac{\dfrac{}{A(y) \vee B(y)}2 \qquad \dfrac{\dfrac{\dfrac{}{A(y)}3}{\exists x A(x)}}{\exists x A(x) \vee \exists x B(x)} \qquad \dfrac{\dfrac{\dfrac{}{B(y)}3}{\exists x B(x)}}{\exists x A(x) \vee \exists x B(x)}}{\exists x A(x) \vee \exists x B(x)}3}{\dfrac{\exists x A(x) \vee \exists x B(x)}{\exists x \, (A(x) \vee B(x)) \Rightarrow \exists x A(x) \vee \exists x B(x)}1}2$$

Recall

$$\dfrac{\exists x P(x) \qquad \begin{array}{c} \dfrac{}{P(y)}1 \\ \vdots \\ Q \end{array}}{Q}$$

What if the domain of quantification is empty?!

$P(x)$: predicate with variable x ranging over $X = \emptyset$.

$$\boxed{\begin{array}{lll} \exists x.\, P(x) & \text{is} & \bot \ \text{(false)} \\ \forall x.\, P(x) & \text{is} & \top \ \text{(true)} \end{array}}$$

But, why?

$$\dfrac{\exists x. A(x) \qquad \begin{array}{c} \overline{A(y)}^{1} \\ \vdots \\ B \end{array}}{B}$$

this holds trivially for any B.

Therefore, $\dfrac{\exists x. A(x)}{B}$

similar to $\dfrac{\bot}{B}$

Q. Can we derive

$$\frac{\forall x.\ A(x)}{\exists x.\ A(x)}$$

$$\cfrac{\cfrac{\forall x.\ A(x)}{A(y)}}{\exists x.\ A(x)}$$

Aristotle thought we can.

But what if the domain/universe of discourse of A is empty? We get

non-sense!

$$\frac{\top}{\bot}$$

Most likely, Aristotle must have excluded empty universes of discourse.

Let's prove

$$(\exists x P(x) \Rightarrow Q) \Leftrightarrow \forall x \, (P(x) \Rightarrow Q)$$

$$\dfrac{\dfrac{\dfrac{\overline{(\exists x P(x) \Rightarrow Q)}^{\,1} \qquad \dfrac{\overline{P(y)}^{\,2}}{\exists y \, P(y)}}{Q}\,(\Rightarrow E)}{\forall x \, (P(x) \Rightarrow Q)}\,(2)}{(\exists x P(x) \Rightarrow Q) \Rightarrow \forall x \, (P(x) \Rightarrow Q)}\,(1)$$

$$\dfrac{\dfrac{\dfrac{\overline{\exists x P(x)}^{\,2} \qquad \dfrac{\overline{P(x)}^{\,3} \qquad \dfrac{\overline{\forall x \, (P(x) \Rightarrow Q)}^{\,1}}{P(x) \Rightarrow Q}}{Q}\,(\Rightarrow E)}{Q}\,(3)\,(\exists E)}{\exists x P(x) \Rightarrow Q}\,(2)}{\forall x \, (P(x) \Rightarrow Q) \Rightarrow \exists x P(x) \Rightarrow Q}$$

Many Sorted Logic

So far, in each example we have considered our variables range over the same universe of discourse.

$\forall x \exists y \; x \leq y$ where x, y are integers.

$\exists x \forall y \; \heartsuit(y, x)$ where x, y are humans.

...

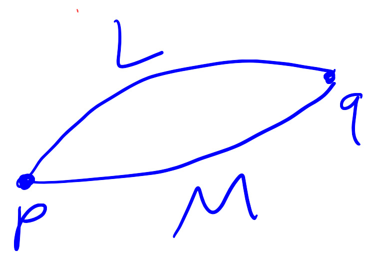
Consider the following example from Euclidean geometry.

Suppose we want to say that for any two <u>distinct</u> points on the Euclidean plane there is a <u>unique</u> line which passes through them.

$$\forall p \forall q \forall L \forall M$$

$$\big(Point(p) \wedge Point(q) \wedge Line(L) \wedge Line(M) \wedge On(p,L) \wedge On(q,L) \wedge On(p,M) \wedge On(q,M)\big) \Rightarrow (p \neq q) \Rightarrow L = M$$

This is very Tedious!



Instead, we introduce **Sorts**.

In the previous example,

Instead of writing

$\mathrm{Point}(p), \mathrm{Point}(q), \ldots$ and

$\mathrm{Line}(L), \mathrm{Line}(M), \ldots$ we

simply introduce two sorts

Point and Line.

When we write $p : \mathrm{Point}$

we mean $\mathrm{Point}(p)$

When we write $L : \mathrm{Line}$

we mean $\mathrm{Line}(L)$.

$$\forall p \forall q \ \forall L \ \forall M$$

$$(\mathrm{Point}(p) \wedge \mathrm{Point}(q) \wedge \mathrm{Line}(L) \wedge \mathrm{Line}(M)$$

$$\wedge \ \mathrm{On}(p, L) \wedge \mathrm{On}(q, L) \wedge \mathrm{On}(p, M) \wedge \mathrm{On}(q, M))$$

$$\Rightarrow (p \neq q) \Rightarrow L = M$$

$$\forall p, q : \mathrm{Point} \quad \forall L, M : \mathrm{Line}$$

$$\mathrm{On}(p, L) \wedge \mathrm{On}(q, L) \wedge \mathrm{On}(p, M), \mathrm{On}(q, M)$$

$$\Rightarrow (p \neq q) \Rightarrow L = M$$

Using equality in logic we can express statements like

(i) there are at least two elements x for which $A(x)$ is true.

$$\exists x,y \ (\neg(x=y) \wedge A(x) \wedge A(y))$$

(ii) there are at most one element x for which $A(x)$ is true.

$$\forall x,y \ (A(x) \wedge A(y) \Rightarrow x=y)$$

(iii) there are at most n elements x for which $A(x)$ is true.

$$\forall x_1, \ldots, x_{n+1} \ (A(x_1) \wedge \ldots \wedge A(x_{n+1}) \Rightarrow$$
$$(x_1 = x_2) \vee \ldots \vee (x_1 = x_{n+1})$$
$$\vee (x_2 = x_3) \vee \ldots \vee (x_2 = x_{n+1})$$
$$\vee \ldots \quad (x_n = x_{n+1}))$$

Special notation for unique existence

We denote the statement that there is a unique x such that $A(x)$ is true by $\exists! x\, A(x)$.

Note that

$$\exists! x\ A(x) \equiv$$

$$\big(\exists x\, A(x)\big) \wedge \big(\forall x \forall y\, (A(x) \wedge A(y) \Rightarrow x = y)\big)$$

$$\equiv$$

$$\exists x\, \big(A(x) \wedge \forall y\, (A(y) \Rightarrow x = y)\big)$$

Counterexamples

Given a formula of the form

$$\forall x.\ A(x)$$

a counterexample is a term t

such that $\neg A(t)$.

Example. Find counterexamples to the statements below

(i) Every prime integer is odd.

(ii) Every integer has a prime factor.

(iii) Every perfect number is even.

$6 = 1 + 2 + 3$

A counterexample is a proof of $\neg\forall x.\, A(x)$ because

$$\neg\forall x.\, A(x) \Longleftrightarrow \exists x.\, \neg A(x)$$

is a tautology, using the law of double negation.

$$
\dfrac{\dfrac{\dfrac{\dfrac{\overline{\neg\forall x.\, A(x)}^{\,1} \quad \dfrac{\dfrac{\dfrac{\dfrac{\overline{\neg\exists x.\neg A(x)}^{\,2} \quad \dfrac{\overline{\neg A(x)}^{\,3}}{\exists x.\neg A(x)}}{\bot}}{\neg\neg A(x)}\,3}{A(x)}}{\forall x.\, A(x)}}{\bot}}{\neg\neg\exists x.\neg A(x)}\,2}{\exists x.\neg A(x)}\,(\mathrm{DN})}{\neg\forall x.\, A(x) \Rightarrow \exists x.\neg A(x)}
$$

Conversely,

$$
\dfrac{\dfrac{}{\exists x.\, \neg A(x)}\,1 \qquad \dfrac{\dfrac{\dfrac{}{\neg A(y)}\,3 \qquad \dfrac{\dfrac{}{\forall x.\, A(x)}\,2}{A(y)}}{\bot}}{\neg \forall x.\, A(x)}\,2}{\dfrac{\neg \forall x.\, A(x)}{\exists x.\, \neg A(x) \Rightarrow \neg \forall x.\, A(x)}\,1}\,3
$$

<u>Exercise</u>. Prove the dual equivalence yourself.

Suppose variable x is not free in B.
We prove the equivalence

$$\exists x\ (A(x) \Rightarrow B) \Longleftrightarrow (\forall x\, A(x) \Rightarrow B)$$

(1) First, let's prove $\exists x\ (A(x) \Rightarrow B) \Rightarrow (\forall x A(x)) \Rightarrow B$

$$\dfrac{\dfrac{\dfrac{}{\exists x\ (A(x) \Rightarrow B)}{}^{1} \qquad \dfrac{\dfrac{\dfrac{}{A(y) \Rightarrow B}{}^{3} \qquad \dfrac{\dfrac{}{\forall x\, A(x)}{}^{2}}{A(y)}}{B}{}^{2}}{\forall x\, A(x) \Rightarrow B}}{\forall x\, A(x) \Rightarrow B}{}^{3}}{\exists x\ (A(x) \Rightarrow B) \Rightarrow (\forall x A(x)) \Rightarrow B}{}^{1}$$

Cor. (Drinker's paradox)

$$\exists x (D(x) \Rightarrow \forall x D(x)) \Longleftarrow (\forall x D(x) \Rightarrow \forall x D(x))$$