MATH 301

INTRODUCTION TO PROOFS

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Fall 2021

- Proof Strategies

Example.

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$$x + \sqrt{x} = 0\,.$$

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\begin{aligned}
& x + \sqrt{x} = 0 & \\
\Rightarrow\ & x = -\sqrt{x} & \text{rearranging} \\
\Rightarrow\ & x^2 = x & \text{squaring} \\
\Rightarrow\ & x(x-1) = 0 & \text{rearranging} \\
\Rightarrow\ & x = 0 \text{ or } x = 1 &
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Now certainly 0 is a solution to the equation, since $0 + \sqrt{0} = 0 + 0 = 0$. However, 1 is *not* a solution, since $1 + \sqrt{1} = 1 + 1 = 2$.

...

Therefore, given a real number x, we have

$$x + \sqrt{x} = 0 \quad \Leftrightarrow \quad x = 0$$

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Note that *the formal expression of our reasoning* is of the form

$$\big((P \vee Q) \wedge \neg Q\big) \Rightarrow P\,.$$

Example

Proposition.

Let $n \in \mathbb{Z}$. Then n^2 leaves a remainder of 0 or 1 when divided by 3.

We use the elimination rule of disjunction (from Definition 1.1.12).

$$\dfrac{p_1 \vee p_2 \vee p_3 \qquad \begin{matrix}[p_1] \\ \rightsquigarrow \\ q\end{matrix} \qquad \begin{matrix}[p_2] \\ \rightsquigarrow \\ q\end{matrix} \qquad \begin{matrix}[p_3] \\ \rightsquigarrow \\ q\end{matrix}}{q} \; (\vee\text{E})$$

Determine what p_1, p_2, p_3 and q are.

Proposition.

Consider the polynomial $p(x) = x^2 + ax + b$ whose coefficients a, b are real numbers and whose discriminant $\Delta = a^2 - 4b$ is non-zero. If $p(x)$ has two distinct roots, then their difference is either a real number or a purely imaginary number. Furthermore, if $\Delta \geqslant 0$, then the difference of the roots is a real number.

First, let us find the logical form of the proposition above.

$$\big(\exists c, d \in \mathbb{R}, \exists \alpha, \beta \in \mathbb{C}, p(\alpha) = 0 \land p(\beta) = 0 \land (\alpha \neq \beta) \land (\alpha - \beta = c + di)\big)$$
$$\Rightarrow \big(c = 0 \lor d = 0\big)$$

OR equivalently,

$$\big(\exists c \in \mathbb{R}, \exists d \in \mathbb{R}, \exists \alpha \in \mathbb{C}, \exists \beta \in \mathbb{C},$$
$$(\alpha^2 + a\alpha + b = 0) \land (\beta^2 + a\beta + b = 0) \land (\alpha \neq \beta) \land (\alpha - \beta = c + di)\big)$$
$$\Rightarrow \big(c = 0 \lor d = 0\big)$$

We use the following proof strategy:

$$\forall x \big(A(x) \Rightarrow B \big) \Rightarrow \exists x A(x) \Rightarrow B$$

That is in order to prove a statement of the form

$$\exists x A(x) \Rightarrow B$$

it suffices to prove

$$\forall x \big(A(x) \Rightarrow B \big)$$

.

Proof.

Let α and β be the (distinct) roots of the polynomial $x^2 + ax + b$. Therefore,

$$x^2 + ax + b = (x - \alpha)(x - \beta)\,.$$

Hence, $\alpha + \beta = -a$ and $\alpha\beta = b$. By the identity $(\alpha + \beta)^2 = (\alpha - \beta)^2 + 2\alpha\beta$, we get $a^2 - 4b = (\alpha - \beta)^2$. The right side of the the latter equation is a real number since the left side is so. Now, suppose $\alpha - \beta = c + di$ for some real numbers c, d. Hence $(\alpha - \beta)^2 = c^2 - d^2 + 2cdi$. Hence, $2cd = 0$ for the right hand side to be a real number. Therefore, either $c = 0$ or $d = 0$. Moreover, if $\Delta \geqslant 0$, by Example 1.1.26, $p(x)$ has two real roots and therefore, their difference is also real. □