Induction

$\mathbb{N} = \{0, 1, 2, 3, \dots\}$

What is our most basic intuition of natural numbers?

① Natural numbers are finitely generated.

(i) $0 \in \mathbb{N}$

(ii) $n \in \mathbb{N} \longmapsto succ(n) \in \mathbb{N}$

Principle of Induction

Suppose P is a predicate on natural numbers.

Suppose

(i) $P(0)$ holds.

(ii) whenever $P(n)$ holds then $P(succ(n))$ holds.

Then $P(n)$ holds for every n.

Proofs by induction are proofs which use the principle of induction.

$$P(0) \wedge (\forall n.\ P(n) \Rightarrow P(succ(n))) \Rightarrow \forall n.\ P(n)$$

$$\dfrac{\begin{array}{c}\vdots\\ P(0)\end{array}\qquad\qquad\begin{array}{c}\overline{P(n)}^{\,1}\\ \vdots\\ P(succ(n))\end{array}}{\forall n.\ P(n)}$$

Thm. For every natural number n,

$$1+2+\cdots+2^n = 2^{n+1}-1$$

Proof. We prove this by induction on n.

When $n=0$,

$$1+2+\cdots+2^n = 2^0 = 1 = 2^1 - 1$$

as required.

For the induction step, fix n and assume that

$$1+2+\cdots+2^n = 2^{n+1} - 1$$

We need to show

$$1+2+\cdots+2^n+2^{n+1} = 2^{(n+1)+1} - 1$$
$$= 2^{n+2} - 1$$

Now, observe that

$$\underline{1+2+\cdots+2^n}+2^{n+1} = \quad \text{(by IH)}$$

$$2^{n+1}-1+2^{n+1} =$$

$$2\cdot 2^{n+1}-1 =$$

$$2^{(n+1)+1}-1$$

Therefore by induction on n, we have that for every natural number n

$$1+\cdots+2^n = 2^{n+1}-1 \quad \square$$

Thm. For every finite set X with n elements the set $\mathcal{P}(X)$ has 2^n elements.

Proof by induction on n.

If $n=0$ then $X=\emptyset$ therefore $\mathcal{P}(X)=\{\emptyset\}$ has $2^0=1$ element

(IH) Suppose for every set with n elements its power set has 2^n elements.

We want to show if X is a set with $n+1$ elements, then $\mathcal{P}(X)$ has 2^{n+1} elements.

Assume X is a set with $n+1$ element. Because $n+1 \geq 1$, X is inhabited. Suppose $a \in X$ for some a.

Any subset S of X either has a or it doesn't.

(i) First, we are counting all subsets S which contain a. There are exactly 2^n of them by the inductive hypothesis (why?)

(ii) Now, we are counting subsets S of X which do not contain a. Again, by IH, there are 2^n of them. (why?)

Therefore by (i), (ii) we have exactly $2^n + 2^n = 2^{n+1}$ subsets of X.

Therefore X has 2^{n+1} subsets as required by the induction step. $\square$

The Least Element Principle

(aka predicate)

Suppose P is some property of natural numbers.

Suppose P holds of some n.

Then there is a smallest value of n for which P holds.

Thm. Assuming LEM, the principle of induction and the principle of least element are equivalent.

Principle of strong induction

P: predicate on $\mathbb{N}$

suppose $P(0)$ holds

and for every $m \geq 1$

$P(n)$ holds for all $n < m$.

Then $P(m)$ holds.

Principle of strong Induction

$\Leftrightarrow$

Induction.

Thm. Every natural number $n \geq 2$ can be factored into a product of prime numbers.

Proof. $n=2$ is prime.

Now suppose $n > 2$.

Either n is prime in which case $n = p$, and we are done,

or n is not prime, in which case $n = m \cdot k$ for some m, k smaller than n.

By the induction hypothesis there are prime numbers $p_1, \ldots, p_\ell$ s.t. $m = p_1 \cdots p_\ell$ (I)

and there are prime numbers $q_1, \ldots, q_s$ s.t. $k = q_1 \cdots q_s$ (II).

From (I) & (II) we have

$n = mk = p_1 \cdots p_\ell q_1 \cdots q_s$ which is a product of primes. $\square$