MATH 301

INTRODUCTION TO PROOFS

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Spring 2022

Natural Deduction for Quantifiers

The inference rules for ∀ and ∃

The introduction rule

$$\frac{A(t)}{\forall y\ A(y)}\ \forall\mathrm{i}$$

The elimination rule

$$\frac{\forall x\ A(x)}{A(t)}\ \forall\mathrm{e}$$

The introduction rule

$$\frac{A(t)}{\exists x A(x)}\ \exists\mathrm{i}$$

The elimination rule

$$\dfrac{\exists x A(x) \qquad \begin{array}{c} \overline{A(y)}^{\,1} \\ \vdots \\ B \end{array}}{B}\ {}^{1}\ \exists\mathrm{e}$$

The Rules of Inference for Universal Quantifier ($\forall$)

Here

- A is a predicate on a domain of discourse X,
- t is a *generic* term of domain X, and
- x and y are variables from domain X.

These rules are valid under two important caveats of the next slides.

The introduction rule

$$\frac{A(t)}{\forall y\ A(y)}\ \forall\mathrm{i}$$

The elimination rule

$$\frac{\forall x\ A(x)}{A(t)}\ \forall\mathrm{e}$$

If Things Can Go Wrong, They Will Go Horribly Wrong, Absurdly!

Note that there are important caveats for $\forall$i and $\forall$e to be valid, and if they are not observed, things can go horribly wrong (i.e. we arrive at absurd conclusions.)

If Things Can Go Wrong, They Will Go Horribly Wrong, Absurdly!

Note that there are important caveats for $\forall$i and $\forall$e to be valid, and if they are not observed, things can go horribly wrong (i.e. we arrive at absurd conclusions.) Define the predicates A and B in one free variable x of natural numbers $\mathbb{N}$ by

$$A(x) := \exists y : \mathbb{N}\ (y < x) \qquad B(x) := \exists y : \mathbb{N}\ (y > x)\,.$$

Be warned that if we apply the rules $\forall$i and $\forall$e naively, we get the following absurd conclusions:

$$\cfrac{\cfrac{\cfrac{\cfrac{\cfrac{\exists n : \mathbb{N}\ (t - 1 = n^2)}{\exists n : \mathbb{N}\ (t = n^2 + 1)}}{1 < t}}{\exists y\ (y < t)}}{A(t)}}{\forall y\ A(y)}\ \forall\mathrm{I} \qquad\qquad \cfrac{\cfrac{\forall x\ B(x)}{B(y+1)}\ \forall\mathrm{e}}{\exists y\ (y > y + 1)}$$

But, What's The Problem?

Here are the problems with the above inferences:

1. In the left-hand side inferences involving $\forall$i, we have more information about t than just $t : \mathbb{R}$. So, t is not generic. (It's like inferring "war is mother of all inventions" and verifying that statement by examining inventions resulted from war time urgency and mobilizations.

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1. In the left-hand side inferences involving $\forall$i, we have more information about t than just $t : \mathbb{R}$. So, t is not generic. (It's like inferring "war is mother of all inventions" and verifying that statement by examining inventions resulted from war time urgency and mobilizations.
2. In the right-hand side inference, the sin we are committing is that the term $t = y + 1$ which we substitute for x in $B(x)$ has the variable y in it which is bound in $B(x)$. Another example of such a sin lead us to infer, incorrectly of course, from the assumption that every child has a parent, the conclusion that every child is a parent of himself or herself (a logical impossibility, at least without traveling back in time).

How To Solve The Inference Problems For ∀i and ∀e

The answer to any problem in logic is always "with more rules".

1. In the introduction rule, t being generic means that it should not have any property other than being a from X, and this means that t should not be free in any uncanceled hypothesis.

The introduction rule

$$\frac{A(t)}{\forall y\ A(y)}\ \forall\mathrm{i}$$

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$$\frac{\forall x\ A(x)}{A(t)}\ \forall\mathrm{e}$$

How To Solve The Inference Problems For ∀i and ∀e

The answer to any problem in logic is always "with more rules".

1. In the introduction rule, t being generic means that it should not have any property other than being a from X, and this means that t should not be free in any uncanceled hypothesis.
2. In the elimination rule, t can be any term that does not clash with any of the bound variables in A.

The introduction rule

$$\frac{A(t)}{\forall y\ A(y)}\ \forall\text{i}$$

The elimination rule

$$\frac{\forall x\ A(x)}{A(t)}\ \forall\text{e}$$

How do we prove an existential statement?

Suppose we want to prove that there exists an odd composite number. We can formalize this statement as

$$\exists n \in \mathbb{N}\ \neg\mathrm{even}(n) \wedge \neg\mathrm{prime}(n)$$

where the predicates even and prime are the predicates of "evenness" and "primeness" of numbers, respectively.

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To prove this statement, we just present a candidate, and show that the candidate satisfies the required properties. For example, we could choose 15, and then show that 15 is not even (by witnessing $15 = 2 \times 7 + 1$) and that 15 is not prime (by witnessing $15 = 3 \times 5$. Of course, there's nothing special about 15, and we could have proven it also using a different number, like 27 or 39. The choice of candidate does not matter, as long as it has the required properties.

In natural deduction this proof appears as:

$$
\dfrac{\dfrac{\dfrac{\vdots}{\neg\mathrm{even}(15)} \qquad \dfrac{\vdots}{\neg\mathrm{prime}(15)}}{\neg\mathrm{even}(15) \wedge \neg\mathrm{prime}(15)}\ (\wedge\mathrm{i})}{\exists n\ (\neg\mathrm{even}(n) \wedge \neg\mathrm{prime}(n))}
$$

Another Example of How We Prove An Existential Statement

Consider the predicate of sexy on prime numbers which we defined in our Lean Lab as a binary predicate

$$\text{sexy} \colon \mathbb{N} \to \mathbb{N} \to \mathbb{P}\text{rop}$$

where

$$\text{sexy } m \ n \ := \text{prime}(m) \wedge \text{prime}(n) \wedge (m - n = 6 \vee n - m = 6)$$

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$$\text{sexy } m \ n \ := \text{prime}(m) \wedge \text{prime}(n) \wedge (m - n = 6 \vee n - m = 6)$$

Suppose we want to prove that there are sexy primes. To do this, we only need to present two candidates, and show that they satisfies the required properties. For example, we could choose 5 and 11, and then show that 5 is prime and 11 is prime and their difference is 6. Of course, there's nothing special about 5 and 11, and we could have proven it also using different candidates, like 13 and 19. For the existential statement

$$\exists m \, \exists n \text{ sexy } m \ n$$

to be true the choices of candidates do not matter.

The Rules of Inference for Existential Quantifier ($\exists$)

Let's make the above observations into rules:

Here

- A, B are predicates on a domain of discourse X,
- t is any term of domain X, and
- x is variables from domain X, and
- y is a generic term of X.

These rules are valid under two important caveats of the next slide.

The introduction rule

$$\frac{A(t)}{\exists x A(x)}\ \exists\mathrm{i}$$

The elimination rule

$$\frac{\exists x A(x) \qquad \begin{matrix} \overline{A(y)}^{\,1} \\ \vdots \\ B \end{matrix}}{B}\ 1\ \exists\mathrm{e}$$

The Intuition Behind The Elimination Rule $\exists e$

If we know $\exists x\, A(x)$, we can temporarily reason about an arbitrary element y satisfying $A(y)$ in order to prove a conclusion that doesn't depend on y.

The Intuition Behind The Elimination Rule ∃e

If we know $\exists x\, A(x)$, we can temporarily reason about an arbitrary element y satisfying $A(y)$ in order to prove a conclusion that doesn't depend on y.

Suppose I have a task G at hand and in general any of various tools, say T_i's, can be used to achieve task G. I have also been given a toolbox but I cannot, for some reason, inspect into it and see which tools it contains. My toolbox may or may not have any of the tools T_i. From the knowledge that my toolbox has some T_i, but not knowing which one, I know that I can achieve G.

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Think about it: How did I know that I can achieve G from the knowledge that my toolbox has some T_i?

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Think about it: How did I know that I can achieve G from the knowledge that my toolbox has some T_i?

Because I knew any T_i is equally as good as any other so far as the achieving task G is concerned.

The Caveats For ∃i and ∃e

The answer to any problem in logic is always "with more rules".

1. In the introduction rule, t must not clash with any of the bound variables in A.

The introduction rule

$$\frac{A(t)}{\exists x A(x)}\ \exists\text{i}$$

The Caveats For ∃i and ∃e

The answer to any problem in logic is always "with more rules".

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Define the predicates A and B in one free variable x of natural numbers $\mathbb{N}$ by

$$A(x) := \forall y \; (y \leqslant x) \qquad B(x) := \forall y \; (x \leqslant y)\,.$$

$$\dfrac{\forall y \; (y \leqslant y)}{\exists x \, \forall y \; (y \leqslant x)} \qquad\qquad \dfrac{\exists x \; B(x) \qquad \dfrac{\dfrac{\dfrac{}{\exists n \; (z^2 = n+1)}\,{}^{1}}{z \geqslant 1} \qquad \dfrac{\dfrac{}{B(z)}\,{}^{2}}{z \leqslant 0}}{\perp}}{\perp}\;{}^{2}\ \exists\mathrm{e}$$

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- In the first derivation we have mistakenly substituted y for t in the inference rule.
- In the second derivation, z was not generic.

Examples of Natural Deduction Proofs Involving Quantifiers

In the following examples of natural deduction proofs identify the rules of inference by writing down the proper line on the right hand side of inference lines:

An Example of Natural Deduction for Quantifiers

Example:

In below we construct a natural deduction proof of

$$\forall x\ (Ax \Rightarrow Bx) \Rightarrow \exists x\ Ax \Rightarrow \exists x\ Bx\,.$$

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To do this end, we do the following steps:

$$\dfrac{}{\forall x\ (Ax \Rightarrow Bx) \Rightarrow \exists x\ Ax \Rightarrow \exists x\ Bx}\ 1\quad (\Rightarrow\text{i})$$

- Introduce hypothesis $\forall x\ (Ax \Rightarrow Bx)$ and annotate it by 1.

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{}{\forall x\ (Ax \Rightarrow Bx)}}{\quad}}{\quad}}{\quad}}{\forall x\ (Ax \Rightarrow Bx) \Rightarrow \exists x\ Ax \Rightarrow \exists x\ Bx}\ 1\quad (\Rightarrow\mathrm{i})$$

1. Introduce hypothesis $\forall x\ (Ax \Rightarrow Bx)$ and annotate it by 1.
2. Introduce hypothesis $\exists x\ Ax$ and annotate it by 2.

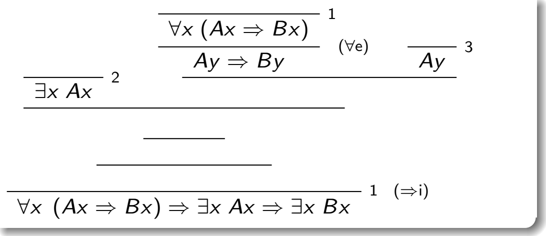
$$
\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{}{\exists x\ Ax}\,{}^{2} \qquad \dfrac{\dfrac{}{\forall x\ (Ax \Rightarrow Bx)}\,{}^{1}}{\quad}}{\quad}}{\quad}}{\quad}}{\forall x\ (Ax \Rightarrow Bx) \Rightarrow \exists x\ Ax \Rightarrow \exists x\ Bx}\ {}^{1}\ (\Rightarrow\mathrm{i})
$$

1. Introduce hypothesis $\forall x\ (Ax \Rightarrow Bx)$ and annotate it by 1.
2. Introduce hypothesis $\exists x\ Ax$ and annotate it by 2.
3. We are trying to eliminate the hypothesis 2; to do that introduce an arbitrary term y which satisfies A.

$$\dfrac{\dfrac{}{\exists x\ Ax}\,2 \qquad \dfrac{\dfrac{}{\forall x\ (Ax \Rightarrow Bx)}\,1 \qquad \dfrac{}{Ay}\,3}{}}{}$$

$$\dfrac{}{\forall x\ (Ax \Rightarrow Bx) \Rightarrow \exists x\ Ax \Rightarrow \exists x\ Bx}\,1\ (\Rightarrow\mathrm{i})$$

1. Introduce hypothesis $\forall x\,(Ax \Rightarrow Bx)$ and annotate it by 1.
2. Introduce hypothesis $\exists x\; Ax$ and annotate it by 2.
3. We are trying to eliminate the hypothesis 2; to do that introduce an arbitrary term y which satisfies A.
4. From $\forall x\,(Ax \Rightarrow Bx)$ we derive $Ay \Rightarrow By$ by $\forall$ elimination.



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5. We apply $\Rightarrow$ elimination to obtain By from $Ay \Rightarrow By$ and Ay.

$$
\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{}{\exists x\ Ax}\,2 \qquad \dfrac{\dfrac{\dfrac{}{\forall x\ (Ax \Rightarrow Bx)}\,1}{Ay \Rightarrow By}\,(\forall e) \qquad \dfrac{}{Ay}\,3}{By}\,(\Rightarrow e)}{\;}}{\;}}{\;}}{\forall x\ (Ax \Rightarrow Bx) \Rightarrow \exists x\ Ax \Rightarrow \exists x\ Bx}\;1\quad(\Rightarrow i)
$$

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6. since y was arbitrary we derive By from $\exists x\ Ax$ and cancel the hypothesis 3.

$$
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$$

$$
\dfrac{\qquad\qquad}{\forall x\ (Ax \Rightarrow Bx) \Rightarrow \exists x\ Ax \Rightarrow \exists x\ Bx}\,1\ (\Rightarrow i)
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7. From By we derive $\exists x\ Bx$ by the introduction rule of $\exists$.

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{}{\exists x\ Ax}\ {}^{2} \qquad \dfrac{\dfrac{\dfrac{}{\forall x\ (Ax \Rightarrow Bx)}\ {}^{1}}{Ay \Rightarrow By}\ (\forall e) \qquad \dfrac{}{Ay}\ {}^{3}}{By}\ (\Rightarrow e)}{By}\ {}^{3}\ (\exists e)}{\exists x\ Bx}\ (\exists i)}{}}{\forall x\ (Ax \Rightarrow Bx) \Rightarrow \exists x\ Ax \Rightarrow \exists x\ Bx}\ {}^{1}\ (\Rightarrow i)$$

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7. From By we derive $\exists x\ Bx$ by the introduction rule of $\exists$.
8. Finally we eliminate hypotheses 2 and 1 by implication introduction.

$$
\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{}{\exists x\ Ax}\,{\scriptstyle 2} \qquad \dfrac{\dfrac{\dfrac{}{\forall x\ (Ax \Rightarrow Bx)}\,{\scriptstyle 1}}{Ay \Rightarrow By}\,{\scriptstyle (\forall e)} \qquad \dfrac{}{Ay}\,{\scriptstyle 3}}{By}\,{\scriptstyle (\Rightarrow e)}}{By}\,{\scriptstyle 3\ (\exists e)}}{\exists x\ Bx}\,{\scriptstyle (\exists i)}}{\exists x\ Ax \Rightarrow \exists x\ Bx}\,{\scriptstyle 2\ (\Rightarrow i)}}{\forall x\ (Ax \Rightarrow Bx) \Rightarrow \exists x\ Ax \Rightarrow \exists x\ Bx}\,{\scriptstyle 1\ (\Rightarrow i)}
$$

Challenge 2:

$$
\dfrac{\dfrac{\dfrac{\dfrac{\overline{\forall x\ A(x)}^{\,1}}{A(y)}}{A(y) \lor B(y)}}{\forall x\ (A(x) \lor B(x))}}{\forall x\ A(x) \Rightarrow \forall x\ (A(x) \lor B(x))}\,1
$$

Challenge 3:

$$
\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\overline{\forall x\, A(x)}\;^{1}}{A(y)} \qquad \dfrac{\overline{\forall x\, B(x)}\;^{2}}{B(y)}}{A(y) \wedge B(y)}}{\forall y\, (A(y) \wedge B(y))}}{\forall x\, B(x) \Rightarrow \forall y\, (A(y) \wedge B(y))}\;^{2}}{\forall x\, A(x) \Rightarrow (\forall x\, B(x) \Rightarrow \forall y\, (A(y) \wedge B(y)))}\;^{1}
$$

Challenge 4:

$$
\dfrac{\dfrac{\,}{\exists x\;(A(x)\wedge B(x))}\;{}^{2}
\qquad
\dfrac{\dfrac{\dfrac{\dfrac{\,}{\forall x\;(A(x)\Rightarrow\neg B(x))}\;{}^{1}}{A(x)\Rightarrow\neg B(x)}
\qquad
\dfrac{\dfrac{\,}{A(x)\wedge B(x)}\;{}^{3}}{A(x)}}{\neg B(x)}
\qquad
\dfrac{\dfrac{\,}{A(x)\wedge B(x)}\;{}^{3}}{B(x)}}{\bot}}
{\dfrac{\dfrac{\bot}{\neg\exists x\;(A(x)\wedge B(x))}\;{}^{2}}{\forall x\;(A(x)\Rightarrow\neg B(x))\Rightarrow\neg\exists x\;(A(x)\wedge B(x))}\;{}^{1}}\;{}^{3}
$$

Challenge 5:

$$
\begin{prooftree}
\AxiomC{}
\RightLabel{1}
\UnaryInfC{$\neg\forall x\ A(x)$}
\AxiomC{}
\RightLabel{4}
\UnaryInfC{$\neg(\exists x\ \neg A(x))$}
\AxiomC{}
\RightLabel{5}
\UnaryInfC{$\neg A(x)$}
\UnaryInfC{$\exists x\ \neg A(x)$}
\BinaryInfC{$\bot$}
\RightLabel{5}
\UnaryInfC{$A(x)$}
\UnaryInfC{$\forall x\ A(x)$}
\BinaryInfC{$\bot$}
\RightLabel{4}
\UnaryInfC{$\exists x\ \neg A(x)$}
\AxiomC{}
\RightLabel{1}
\UnaryInfC{$\exists x\ \neg A(x)$}
\AxiomC{}
\RightLabel{3}
\UnaryInfC{$\neg A(y)$}
\AxiomC{}
\RightLabel{2}
\UnaryInfC{$\forall x\ A(x)$}
\UnaryInfC{$A(y)$}
\BinaryInfC{$\bot$}
\RightLabel{3}
\BinaryInfC{$\bot$}
\RightLabel{2}
\UnaryInfC{$\neg\forall x\ A(x)$}
\RightLabel{1}
\BinaryInfC{$\neg\forall x\ A(x) \leftrightarrow \exists x\ \neg A(x)$}
\end{prooftree}
$$

Challenge 6:

$$\dfrac{\dfrac{\dfrac{}{\exists x\,(A(x)\vee B(x))}\,1 \qquad \dfrac{\dfrac{}{A(y)\vee B(y)}\,2 \qquad \dfrac{\dfrac{\dfrac{}{A(y)}\,3}{\exists x\,A(x)}}{\exists x\,A(x)\vee\exists x\,B(x)} \qquad \dfrac{\dfrac{\dfrac{}{B(y)}\,3}{\exists x\,B(x)}}{\exists x\,A(x)\vee\exists x\,B(x)}}{\exists x\,A(x)\vee\exists x\,B(x)}\,3}{\exists x\,A(x)\vee\exists x\,B(x)}\,2}{\exists x\,(A(x)\vee B(x))\Rightarrow\exists x\,A(x)\vee\exists x\,B(x))}\,1$$