

MATH 301

INTRODUCTION TO PROOFS

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Spring 2022

First Week (Summary)

Overview

- ① Connectives
- ② Rule of The Game (of Inference)
- ③ Examples of Natural Deduction Proof Trees

New Propositions From The Old

- Given propositions P and Q , we can make the following new propositions:

Proposition	Notation
P and Q	$P \wedge Q$
P or Q	$P \vee Q$
P implies Q	$P \Rightarrow Q$
not P	$\neg P$
P if and only if Q	$P \Leftrightarrow Q$

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P if and only if Q	$P \Leftrightarrow Q$

- Note that $P \Leftrightarrow Q$ is defined to be

$$(P \Rightarrow Q) \wedge (Q \Rightarrow P)$$

Few Things to Note

- Note that we use upper-case letters to denote propositions.
- $P \Rightarrow Q$: if P then Q , or P is sufficient for Q , or Q is necessary from P .
- $\neg P$: it is not the case that P .
- A propositional formula is built from propositional atoms (aka variables) and logical operators: e.g. $P \wedge (\neg Q \Rightarrow R) \vee (\neg P \Rightarrow \neg(R \vee S))$

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- If Q can be derived from the assumption that P is true, then $P \Rightarrow Q$ is true;

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$P \Rightarrow Q$ represents the expression “if P , then Q ”.

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$P \wedge Q$ represents “ P and Q ”.

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$$\frac{\begin{array}{cc} \overline{P}^1 & \overline{Q}^1 \\ \vdots & \vdots \\ P \vee Q & R \end{array}}{R} \text{1 ve}$$

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- If P is true, then $P \vee Q$ is true;
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- If $P \vee Q$ is true, and if R can be derived from P and from Q , then R is true.

$P \vee Q$ represents “ P or Q ”.

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The expression $\neg P$ represents “not P ” (or “ P is false”).

The elimination rule

$$\frac{\perp}{P} \perp e$$

The Rules of Inference for Negation

The **negation** operator is the logical operator \neg , defined according to the following rules:

- If a contradiction can be derived from the assumption that P is true, then $\neg P$ is true;
- If $\neg P$ and P are both true, then a contradiction may be derived.

The expression $\neg p$ represents “not P ” (or “ P is false”).

The introduction rule

$$\frac{\begin{array}{c} \overline{P}^1 \\ \vdots \\ \perp \end{array}}{\neg P}^1 \neg i$$

The elimination rule

$$\frac{\neg P \quad P}{\perp} \neg e$$

In order to prove a proposition P is false (that is, that $\neg P$ is true), it suffices to assume that P is true and derive a contradiction.

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We construct a proof of

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from no hypotheses and thereby establish the formula above as a tautology.

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Currying

In steps, we show that

$$(P \Rightarrow (Q \Rightarrow R)) \Rightarrow (P \wedge Q \Rightarrow R)$$

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Eliminating Cases

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$$\frac{\frac{\frac{[((P \vee Q) \wedge \neg Q)]^1}{P \vee Q} \quad \frac{\overline{P}^2}{P} \quad \frac{\frac{\frac{\overline{Q}^2}{Q} \quad \frac{\frac{(P \vee Q) \wedge \neg Q^1}{\neg Q}}{\perp}}{P}^2}{(P \vee Q) \wedge \neg Q \Rightarrow P}^1$$

Challenge: Add annotations for inference rules (on the right side of each horizontal inference line) in the proof tree above.

Conjunction Distributes over Disjunction

We prove in below that the propositional formula

$$(P \wedge (Q \vee R)) \Rightarrow ((P \wedge Q) \vee (P \wedge R))$$

is a tautology.

$$\frac{\frac{\frac{P \wedge (Q \vee R)}{Q \vee R}^1 \quad \frac{\frac{\frac{P \wedge (Q \vee R)}{P}^1 \quad \overline{Q}^2}{P \wedge Q} \quad \frac{\frac{\frac{P \wedge (Q \vee R)}{P}^1 \quad \overline{R}^2}{P \wedge R}}{(P \wedge Q) \vee (P \wedge R)}^2}{(P \wedge Q) \vee (P \wedge R)}^2}{(P \wedge (Q \vee R)) \Rightarrow ((P \wedge Q) \vee (P \wedge R))}^1$$

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Proof by Contrapositive

We prove the proposition $\neg Q \Rightarrow \neg P$ from the assumption $P \Rightarrow Q$.

This is usually referred to as proof by contrapositive; if we know that Q follows from P , and yet Q is not the case, then P is also not the case (If it was, Q would be too).

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$$\frac{\frac{\overline{\neg Q}^1}{Q \Rightarrow \perp} \quad \frac{\frac{\overline{P}^2 \quad P \Rightarrow Q}{Q} (\Rightarrow e)}{\perp} (\Rightarrow e)}{\frac{P \Rightarrow \perp}{\neg P}^2 (\Rightarrow i)} (\Rightarrow i)$$

If P Then Not Not P

We prove that every proposition implies its double negation, that is for every proposition P , the formula

$$P \Rightarrow \neg\neg P$$

is a tautology.

$$\frac{\frac{\frac{\overline{P}^1 \quad \overline{\neg P}^2}{\perp}}{\neg\neg P}^2 \text{ } (\Rightarrow e)}{P \Rightarrow \neg\neg P}^1 \text{ } (\Rightarrow e)$$