MATH 301

INTRODUCTION TO PROOFS

Sina Hazratpour
Johns Hopkins University
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- sets from logic
- operations on sets
- power sets

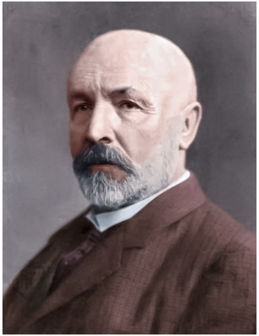
Relevant sections of the textbook

- Chapter 2

Set theory is the theory of everything!

- Set theory is a foundation for mathematics. This means
  1. All abstract mathematical concepts can be expressed in the language of set theory.
  2. All concrete mathematical objects can be encoded as sets.

“ By a set we mean any collection *M* of determinate, distinct objects (called the elements of *M*) of our intuition or thought into a whole.”
(Georg Cantor, 1985)



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- The collection of everything in the universe of discourse is called the universal set, denoted by $\mathcal{U}$.

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- Note that the predicate P can have many variables.

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4. $\{x \mid c(x) = \texttt{yellow} \wedge s(x) = \texttt{triangle}\} = \emptyset = \{\}$

Instead of

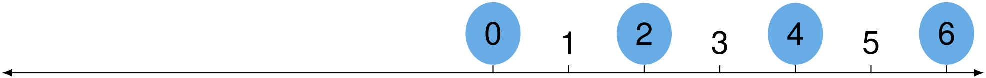
$$\sout{E = \{2, 4, 6, \ldots\}}$$

we use

$$E = \{n \in \mathbb{N} \mid n \text{ is even}\}.$$

More formally, this set is written as

$$\{n \in \mathbb{N} \mid \exists k \in \mathbb{N},\ n = 2k\}.$$



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- $\{n \in \mathbb{N} \mid n \text{ can be written as a sum of its proper divisors}\}$
- $\{a \in \mathbb{R} \mid a \text{ is equal to } 1, 2, 3, \text{ or } \pi\}$

An alternative to set-builder notation

An alternate form of set-builder notation uses an expression involving one or more variables to the left of the vertical bar, and the range of the variable(s) to the right. The elements of the set are then the values of the expression as the variable(s) vary:

$$\{\text{expr}(x) \mid x \in X\} \text{ is defined to mean } \{y \mid \exists x \in X,\ y = \text{expr}(x)\}$$

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Example

The expression $\{2n \mid n \in \mathbb{N}\}$ *denotes the set of even numbers. It is shorthand for* $\{n \in \mathbb{N} \mid \exists k \in \mathbb{N},\ n = 2k\}$*.*

Example

We can use a mix of the two notations:

$$\{p^2 + 1 \mid p \text{ is prime}\}\,.$$

Some important sets

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- For an object a, we have $\{x \in \mathcal{U} \mid x = a\}$ is the singleton set $\{a\}$.
- For distinct objects a and b, we have $\{x \in \mathcal{U} \mid (x = a) \vee (x = b)\}$ is the set $\{a, b\}$.

Inhabited vs non-empty

A set X is inhabited if it has at least one element. Formally, a set X is inhabited if the sentence

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– or equivalently the sentence $\exists x \, (x \in X)$ – is true.

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Exercise

Use natural deduction to show that $\emptyset$ is empty.

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Use natural deduction to show that every inhabited set is non-empty.

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Relative complement $X \setminus Y = \{x \in X \mid x \notin Y\} =_{\text{def}} \{x \mid (x \in X) \wedge \neg(x \in Y)\}$

Logical operations and set operations

The important sets and operations we have built so far are readily representable in symbolic logic.

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Equality of sets

1. Are the sets

$$\{n \in \mathbb{N} \mid \exists k \in \mathbb{N},\ n = 2k\} \quad \text{and} \quad \{n \in \mathbb{Q} \mid \exists k \in \mathbb{N},\ n = 2k\}$$

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2. How about 'the set of prime numbers less than 2' and 'the set of even prime numbers greater than 2'?

3. How about

$$\{x \in \mathbb{Q} \mid x^2 < 2\} \quad \text{and} \quad \{x \in \mathbb{Q} \mid x^2 \leqslant 2\}\,?$$

Extensional equality of sets

Definition (Set extensionality)

Two sets A and B are equal precisely when they have the same elements.

The formal sentence expressing $A = B$ is

$$\forall x\,(x \in A \Leftrightarrow x \in B)\,.$$

Therefore, using the extensional definition of equality of sets, the answers to the questions (1)-(3) of the previous slide are all positive.

As an exercise we prove the distributivity of intersection ($\cap$) over union ($\cup$) of sets.

Theorem

Let A, B, and C denote sets of elements of some domain. Then $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$.

Proof.

Let x be arbitrary, and suppose x is in $A \cap (B \cup C)$. Then x is in A, and either x is in B or x is in C. In the first case, x is in A and x is in B, and hence x is in $A \cap B$. In the second case, x is in A and C, and hence x is in $A \cap C$. Therefore, x is in $(A \cap B) \cup (A \cap C)$.

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First, suppose x is in $A \cap B$. Then x is in both A and B. Since x is in B, it is also in $B \cup C$, and so x is in $A \cap (B \cup C)$.

The second case is similar: suppose x is in $A \cap C$. Then x is in both A and C, and so also in $B \cup C$. Hence, in this case also, x is in $A \cap (B \cup C)$, as required. □

You should be able to see elements of natural deduction implicitly in the proof above. Explicitly, we need to construct a natural deduction proof of the sentence

$$\forall x \, (x \in A \cap (B \cup C) \leftrightarrow x \in (A \cap B) \cup (A \cap C)) \, .$$

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Subsets

Definition

If A and B are sets, A is said to be a **subset** of B, written $A \subseteq B$, if every element of A is an element of B.

Formally, $A \subseteq B$ is expressed by the sentence

$$\forall x \; (x \in A \Rightarrow x \in B)$$

Exercise

Prove that $A = B$ if and only if $A \subseteq B$ and $B \subseteq A$.

Subsets (II)

Let's prove few facts about the subset relationship:

Exercise

1. *Prove that for all sets A we have $A \subseteq A$.*
2. *Prove that for all sets A, B and C, if $A \subseteq B$ and $B \subseteq C$ then $A \subseteq C$.*
3. *Prove that for all sets A we have $\emptyset \subseteq A$.*
4. *Prove that for all sets A, B, if $A \cup B = B$ then $A \subseteq B$.*
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Remark

It is true that $\varnothing \subseteq \varnothing$*, but false that* $\varnothing \in \varnothing$*. Indeed,*

- $\varnothing \subseteq \varnothing$ *means* $\forall x \in \varnothing,\ x \in \varnothing$*; but propositions of the form* $\forall x \in \varnothing,\ p(x)$ *are always true.*
- *The empty set has no elements; if* $\varnothing \in \varnothing$ *were true, it would mean that* $\varnothing$ *had an element (that element being* $\varnothing$*). So it must be the case that* $\varnothing \notin \varnothing$*.*

$$A \cup A^c = \mathcal{U} \qquad A \cap A^c = \emptyset$$

$$A \cup A = A \qquad A \cap A = A$$

$$A \cup \emptyset = A \qquad A \cap \emptyset = \emptyset$$

$$A \cup \mathcal{U} = \mathcal{U} \qquad A \cap \mathcal{U} = A$$

$$A \cup B = B \cup A \qquad A \cap B = B \cap A$$

$$(A \cup B) \cup C = A \cup (B \cup C) \qquad (A \cap B) \cap C = A \cap (B \cap C)$$

$$(A \cup B)^c \subset A^c \cap B^c \qquad (A \cap B)^c \supseteq A^c \cup B^c$$

and

$$A \cap (B \cup C) = (A \cap B) \cup (A \cap C) \qquad A \cap (A \cup B) = A$$

$$A \cup (B \cap C) = (A \cup B) \cap (A \cup C) \qquad A \cup (A \cap B) = A$$

Classical sets

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Exercise

Show that if A is a classical set then $A^{cc} = A$.

A digression: numbers from sets

We can define "fake" numbers by way of sets:

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Are any of these sets satisfactory definitions of natural numbers?

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If I is a set, we will sometimes wish to consider a family $\{A_i \mid i \in I\}$ of sets indexed by elements of I. An alternative notation for a family that we ocassionally use is $(A_i)_{i \in I}$.

For example, we might be interested in a sequence

$$A_0, A_1, A_2, \ldots$$

of sets indexed by the natural numbers.

Example

- *For each natural number n, we can define the set A_n to be the set of people alive today that are of age n.*
- *For every positive real number r we can define B_r to be the interval $[-r, r]$. Then $(B_r)_{r \in \mathbb{R}}$ is a family of sets indexed by the real numbers.*
- *For every natural number n we can define $C_n = \{k \in \mathbb{N} \mid k \text{ is a divisor of } n\}$ as the set of divisors of n.*

Union and intersection of indexed families

Given a family $\{A_i \mid i \in I\}$ of sets indexed by I, we can form its union:

$$\bigcup_{i \in I} A_i = \{x \mid x \in A_i \text{ for some } i \in I\}$$

We can also form the intersection of a family of sets:

$$\bigcap_{i \in I} A_i = \{x \mid x \in A_i \text{ for every } i \in I\}$$

So an element x is in $\bigcup_{i \in I} A_i$ if and only if x is in A_i for *some* i in I,

and

x is in $\bigcap_{i \in I} A_i$ if and only if x is in A_i for every i in I.

These operations are represented in symbolic logic by the existential and the universal quantifiers. We have:

$$\forall x \ (x \in \bigcup_{i \in I} A_i \leftrightarrow \exists i \in I \ (x \in A_i))$$

$$\forall x \ (x \in \bigcap_{i \in I} A_i \leftrightarrow \forall i \in I \ (x \in A_i))$$

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$$\bigcup_{i \in I} A_i = A_0 \cup A_1.$$
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Exercise

What is $\bigcup_{i \in I} A_i$ *and* $\bigcap_{i \in I} A_i$ *when the indexing set* I *is empty?*

Exercise

Prove the following equality of sets:

$$\bigcup_{i \in I} \{i\} = I$$

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Prove the following equalities of sets:

1. $A \cap \bigcup_{i \in I} B_i = \bigcup_{i \in I} (A \cap B_i)$
2. $A \cup \bigcap_{i \in I} B_i = \bigcap_{i \in I} (A \cup B_i)$

We can have a family of sets indexed by many sets: for instance, a family $(A_{i,j})_{i \in I, j \in J}$.

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For every such family, consider the family $(B_i)_{i \in I}$ where $B_i = \bigcup_{j \in J} A_{i,j}$ (fix $i \in I$, and let j range over J). We define $\bigcup_{i \in I} \bigcup_{j \in J} A_{i,j}$ to be $\bigcup_{i \in I} B_i$.

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Exercise

Prove the following equalities of sets:

1. $\bigcup_{i\in I}\bigcup_{j\in J} A_{i,j} = \bigcup_{j\in J}\bigcup_{i\in I} A_{i,j}$
2. $\bigcap_{i\in I}\bigcap_{j\in J} A_{i,j} = \bigcap_{j\in J}\bigcap_{i\in I} A_{i,j}$

Exercise

Show that

$$\bigcup_{i\in I}\bigcap_{j\in J} A_{i,j} \subseteq \bigcap_{j\in J}\bigcup_{i\in I} A_{i,j}$$

Proof.

Let x be an arbitrary member of $\bigcup_{i\in I}\bigcap_{j\in J} A_{i,j}$. Therefore, there is some i, say i_0, such that $x \in \bigcap_{j\in J} A_{i_0,j}$. Therefore for every $j \in J$, $x \in A_{i_0,j}$. Hence, for every $j \in J$ there is some i, namely i_0, such that $x \in A_{i,j}$. Therefore, $x \in \bigcup_{i\in I}\bigcap_{j\in J} A_{i,j}$.

It follows that $\bigcup_{i\in I}\bigcap_{j\in J} A_{i,j} \subseteq \bigcap_{j\in J}\bigcup_{i\in I} A_{i,j}$. □

Exercise

Find the indexing sets I and J and family $(A_{i,j})_{i\in I, j\in J}$ such that

$$\bigcap_{j\in J}\bigcup_{i\in I} A_{i,j} \nsubseteq \bigcup_{i\in I}\bigcap_{j\in J} A_{i,j}$$

Exercise

Find the indexing sets I and J and family $(A_{i,j})_{i \in I, j \in J}$ such that

$$\bigcap_{j \in J} \bigcup_{i \in I} A_{i,j} \not\subseteq \bigcup_{i \in I} \bigcap_{j \in J} A_{i,j}$$

Take the indexing sets I and J to be the set of natural numbers and let $A_{i,j}$ to be the empty set if $i \neq j$, and the singleton set $\{*\}$ if $i = j$.

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Take the indexing sets I and J to be the set of natural numbers and let $A_{i,j}$ to be the empty set if $i \neq j$, and the singleton set $\{*\}$ if $i = j$. Now,

$$\bigcap_{j\in J}\bigcup_{i\in I} A_{i,j} = \{*\}$$

whereas

$$\bigcup_{i\in I}\bigcap_{j\in J} A_{i,j} = \emptyset .$$

The power set

Let X be a set. The power set of X, written $\mathcal{P}(X)$ is the set of all subsets of X.

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Note that the power set of every set is inhabited since for a set X, we have $\varnothing \in \mathcal{P}(X)$ and $X \in \mathcal{P}(X)$.

Finite power sets

Theorem

For any finite set A, if A has n elements, then there are 2^n subsets of A.

Proof.

We use induction on n. In the base case, there is only one set with 0 elements, the empty set, and there is exactly one subset of the empty set, as required.

In the inductive case, suppose A has $n + 1$ elements. Let a be any element of A, and consider the set $A \setminus \{a\}$ be the set containing the remaining n elements. In order to count the subsets of A, we divide them into two groups.

First, we consider the subsets of A that don't contain a. These are exactly the subsets of $A \setminus \{a\}$ and by the inductive hypothesis, there are 2^n of those.

Next we consider the subsets of A that contain a. Each of these is obtained by choosing a subset of $A \setminus \{a\}$ and adding a. Since there are 2^n subsets of $A \setminus \{a\}$, there are 2^n subsets of A that contain a.

Taken together, then, there are $2^n + 2^n = 2^{n+1}$ subsets of A, as required. □

Example

Let X be a set. Define the family $(S_x)_{x \in X}$ where S_x is the set of all subsets of X which contain x. In other words:

$$S_x = \{A \subseteq X \mid x \in A\}.$$

Show that

1. $\displaystyle\bigcup_{x \in X} S_x = \mathcal{P}(X) \setminus \{\emptyset\}$
2. $\displaystyle\bigcap_{x \in X} S_x = \{X\}$

Cartesian product of sets

With the tools we have developed we can define the cartesian product $A \times B$ of sets A and B to be the set containing exactly ordered pairs

$$(a, b) =_{\text{def}} \{\{a\}, \{a, b\}\} \in \mathcal{P}(\mathcal{P}(A \cup B))$$

where $a \in A$ and $b \in B$.
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$$A \times B := \{(a, b) \;\mid a \; \in A \text{ and } b \in B\}\,.$$

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Notice that if $a = b$, the set (a, b) has only one element:

$$(a, a) = \{\{a\}, \{a, a\}\} = \{\{a\}, \{a\}\} = \{\{a\}\}.$$

The following theorem shows that the definition of cartesian product of sets is reasonable.

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$(a, b) = (c, d)$ *if and only if* $a = c$ *and* $b = d$.

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We leave the proof to the reader as an exercise.